Mathematics and Architecture since 1960

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“Mathematics and Architecture since 1960” — an impossibly tall order! In a short paper, I can only sketch an outline of work that I have had direct involvement with since the 1960s. I will touch upon some of my own work and work by some of my closest colleagues in architecture and urban studies.¹ Much of this work is recorded in the academic research journal Environment and Planning B, of which I was appointed founding editor in 1974, and which, under the banner Planning and Design, is now in its 29th year.² Many Environment and Planning B contributors have been colleagues “at a distance” whom I may have met on occasions, or not.³

Introduction

A crude, but useful, distinction between mathematics and architecture is that the former tends towards abstract generalizations, whereas the latter is concretely particular.

In mathematics, take the simple rule of the so-called Pythagorean triangle – in a right triangle the squares on the sides sum to the square on the hypotenuse. Typical of the mathematical enterprise, the mathematician Pappus of Alexandria extended the rule to any triangle with any parallelograms on two sides (Figure 1):

![Figure 1. Pappus’s generalization of the Pythagoras Theorem where, for any triangle, the areas of the arbitrary black parallelograms on the left sum to the area of the appropriately constructed black parallelogram on the right](image)

It is not difficult to see that Pythagoras’s Theorem is just a very special case of Pappus. It is only a step further to arrive at what is known today as the cosine law.

Architecture tends to work the other way round. Given very general notions about buildings, the architect is asked to tease out a specific design to satisfy certain performance expectations. Mathematics can play a part at each of these three level: generic knowledge, specific design, and the prediction of performance. Mathematics has traditionally been used to predict physical performance such as structural, acoustic, thermal, lighting and so on. Here, accepted laws of physics are applied to specific geometries and materials to derive expected results which do, or do not, satisfy requirements. Armed with this

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information, the architects may modify their designs to seek improved performance. Similarly, there is economic modeling in which various costs and benefits are assessed. I shall not dwell on this analytical use of mathematical modeling which has nevertheless developed at a pace over the past forty years. Instead, I shall concentrate on the role that mathematical thought plays in furthering generic knowledge in architecture, and then touch upon its potential use in generating specific designs.

I mean by “generic knowledge” notions generally entertained by architects and the public: the cultural fix, the conventional wisdom. In the 1950s, in the era of reconstruction following WWII, it was generally accepted that tall buildings would make better use of land than the low structures they were to replace. In the early 1960s, I was invited by Sir Leslie Martin to assist in producing a plan for such a reconstruction of Whitehall, the national and government center in London. The government architects had already built a prototypical office complex some mile or so away. It sported three twenty-story towers over a three-story podium. It seemed that the government’s intention was to pull down several Victorian structures (including Gilbert Scott’s “battle of the styles” Foreign Office) and insert new buildings of this type next to Westminster Abbey, Inigo Jones’s Banqueting Hall, William Kent’s Horse Guards, and Norman Shaw’s New Scotland Yard.

Consider an exercise using Friedrich Froebel’s third gift from the wooden construction toys made famous among architects in Frank Lloyd Wright’s *An Autobiography.* A square table marked with twenty-five squares is the site. Placing all eight cubes on the center square creates a high-rise tower; placing them on the eight squares of the next ring creates a low-rise court form. Now the fourth gift has the same volume as the third, but is made from eight bricks with sides in the proportion 4 : 2 : 1. These eight bricks may be arranged around the sixteen perimeter squares at half the height of the last ring and one-sixteenth the height of the central tower (Figure 2).

![Figure 2. Froebelian demonstration showing a given volume distributed in three different ways suggestive of tower and court forms](image)

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Buildings were arranged around courtyards in traditional development in, and projects for, the Whitehall area. London itself is renowned for its Georgian squares – examples of the courtyard form in urban design. I developed a program to explore the possibilities of arranging the built forms (as our very abstract representations of buildings were known) in three distinct arrays: the pavilion form (or isolated structure); the street form; and the court form.

The government was looking to put a certain number of civil servants to work in Whitehall. In building terms this translated into a gross building volume that was to be distributed over the site with a variety of options regarding conservation. The result of computations convinced us that the whole scheme could be arranged in courts surrounded by buildings no higher than the existing Victorian and Edwardian office buildings in the area. Tall towers were not necessary, although even after the plan was accepted the government’s architect pushed on with plans to build an office tower that would have dwarfed Big Ben in its bulk. It seemed that building something to blend in with the existing fabric was not appreciated at that time as a means to express the power of the “modern” state (Figure 3).

These conclusions had been reached by number-crunching on an early card-hungry computer, and represent one of the first uses of electronic computing in architectural design. Shortly thereafter Leslie Martin established the research centre for Land Use and Built Form Studies (LUBFS) initially with funding from the Gulbenkian Foundation, and with myself as Director.  

Forms of knowledge: quantitative studies

One of my first pieces of research was to formalize the results of the Whitehall study. Using no more than high school calculus, the new models demonstrated the superior effectiveness of low-rise courtyard development over street and pavilion forms. These results were then compared to the famous Heilgenthal-Gropius result that underwrote the internationalists’ images of a city of residential slabs and towers. It was concluded that a more discerning appreciation of the mathematical model’s ‘structure’ might have convinced Gropius that the greatest gains were to be found in flachbau, low-rise housing.  

Of course, every mathematical model abstracts from actuality and only deals with a
limited number of factors and assumptions. In my view, such models are useful in questioning our prejudices and sharpening our understanding as long as the limitations are taken fully into account.

A more general principle was extracted from such studies which we named after Fresnel, the French physicist, who had used such a pattern for light refraction gratings. Subsequently we learned that both Leonardo da Vinci and Albrecht Dürer had drawn related diagrams in their graphic studies of isoperimetric problems. Generically, the diagram shows a set of similar plane figures with areas equal to 1, 2, 3, 4, ... respectively. Fresnel and Leonardo used circles, we used squares. When the squares are placed about a single center, the diagram shows a series of concentric rings each of which is of unit area and therefore equal to the area of the central square (Figure 4).

![Diagram of Fresnel square and related spatial distributions](image)

**Figure 4.** Left, the Fresnel square. Center, different spatial distributions of the same content (the density of black against white) from left to right, concentrated to dispersed. Center top row, nucleated distribution. Center bottom row, reticulated distribution. Right, Leonardo da Vinci’s diagram reconstructed from *Codex Atlanticus* 221v-b in which each crescent has the same area as the full circle.

In this diagram, the unit square in the center occupies one-ninth of the largest square. The side lengths are thus proportionately $\sqrt{1} = 1 : \sqrt{2} : \sqrt{3}, \sqrt{4} = 2 : \sqrt{5} : \sqrt{6} : \sqrt{7} : \sqrt{8} : \sqrt{9} = 3$. The illustration shows this same unit area distributed in two ways – nucleated at the center and reticulated around the perimeter. Then, retaining the overall content, the area is fractured into smaller parts, but retaining the overall content. This diagram illustrates a problem with urban designers’ use of the term density, and its subsequent architectural implication that high density implies tall buildings. Spatially there are four distinctions to be made: concentrated-nuclear, concentrated-linear, dispersed-nuclear, dispersed-linear. All have the same density, but exhibit quite distinct spatial distributions that carry quite different architectural and urban design implications.

Much of twentieth-century architecture has a basis in density calculations. Think only of the work of CIAM members in the pre-WWII years and the application of their doctrines in the years of reconstruction that followed. There was much “scientific” hoodwinking concerning persons per square kilometer, bed-spaces per hectare and similar quantifications. This was the language of land-owners, public and private, who sought to
make efficient or profitable use of land. The individual occupant is not put first. We very rarely hear about land per person, per individual.

If land were divided up equally, how much land do different people have around the world? Politically, this is a key question, because others have decided, or are deciding, how to parcel up that land. In Singapore, the Peoples Action Party clearly has decided to give over a substantial part of each individual’s entitlement of land to public open space rather than private patios and gardens. Persons per hectare disguise this reality, whereas starting from the premise that each individual Singaporean – man, woman and child – might count the land of a single’s tennis court as their own, the question might be asked, as a family looks out from its tenth-story apartment, how has most of it been used, for whom and by whom?

In the world today, each individual’s portion of the land surface of the globe (excepting Antarctica) is equivalent to 150 x 150 = 22,500 sq. m. Estimates suggest that at the very worst this might fall to 13,500 sq. m. by 2050, but is far more likely to have levelled out at 19,500 sq. m., or the size of two soccer fields. Does this individual know that some two-thirds of this land is already consumed in permanent pasture, forest, crop and arable land? Or that only two percent of the land surface is actually urbanized? Or that this urban use may barely increase to three percent of the land surface by 2050? Nevertheless, it does beg the question: how will this additional one percent – 1,360,000 sq. km. – be urbanized, a land mass equivalent to the Northern Territory, Australia? Or, if everyone of the estimated 7 billion persons in 2050 were to live at the density of sprawling Los Angeles, that no more land than in use today would be required for urban uses, and that an area the size of Japan might be returned back to non-urban uses? On average, each and every person in Los Angeles enjoys, appropriately, a basketball court of land. Yet, each individual making up the world’s urban population (some 2.82 billion) average two and two-thirds times the land an Angeleno currently has (Figure 5). 8

Figure 5. Distribution of land in different communities and groups in the world population, 2000. The circle with Francesco di Giorgio’s Vitruvian man marks the scale.
I trust these simple arithmetical manipulations and interpretations of publicly-available data challenge the conventional wisdom. Part of the problem is to visualize the arithmetic in sensible ways, and part is that the numbers arise from the shape of urban and architectural artifacts. 2,750 persons per sq. km is an abstract concept: one person standing in an otherwise empty basketball court is concrete. Yet, they are equivalent.

Architecture, in its applications, demands the concretisation of abstract mathematical statements

A study prepared for an The Open University television program in urban geography is suggestive. I posed the problem of the compact city form and showed that without using more land other forms were possible. Take, for example, a nine block section of a theoretical compact city. The nine blocks might be reconfigured spatially as annulated form in which the depth of "service" from the roads is the same as in the compact city; or, as a cruciform. In each of these two reconfigurations the length of road is reduced by a factor of 55%, which means that utility runs (water, sewers, electricity, telephone and cable) that normally follow roads are also reduced by this same amount. The maximum trip length across the compact section is 120% of the same distances across the annular and cruciform configurations. Moreover, both these latter forms show lower expected mean distances than the compact city form: The cruciform shows that trips are likely to be 95% of the compact from, while the ring is best with 92%. Reductions this order are significant in energy terms as well as in convenience. The study suggests compelling counter-examples to the conventional wisdom about city forms (Figure 6).

![Figure 6. Top left, a nine block section of a theoretical compact city; Top center, the same land area reconfigured as an annulated form; Top right, the same again reconfigured as a cruciform; Below, the frequency distribution of trip lengths](image)

Suppose now that a new urban area is to be planned. Conventional surveys would pick out a piece of land and define a closed boundary around the urban area. But why not
mark the protected open areas and let the urban form fill the interstitial spaces? The second two configurations suggest such an approach (Figure 7).

![Figure 7](image)

Figure 7. Left, a theoretical compact city surrounded by a green belt. Center, four ring cities occupying the same amount of urban land. Right, four equivalent cruciform cities unite to create a reticular pattern.

There is no variety in these examples such as central city, market towns and villages which make up many urban regions. Another diagram indicates a “line” alternative to conventional “blob” thinking.

![Figure 8](image)

Figure 8. Left, a theoretical urban region comprising a central city surrounded by a greenbelt, satellite market towns with their greenbelts, and necklaces of villages around each of these. Right, an alternative possibility: a regional park defined in the center, satellite community parks, and smaller village greens. The urbanization is linearized around these protected open areas, occupying the same amount of land as in the conventional configuration.

In the nineteenth century, the Spanish engineer, Ildefons Cerdà (1815-1876), set out the first modern theory of urbanización with the compelling maxim “Rurizad lo urbanos, urbanizad lo rural … replete terram”.

There is an 1861 sketch of his that illustrates the fundamental urban problem with compact city forms: that traffic requires more land as centers are approached at the same time that buildings require more space. In 1961 – quite unknowingly at the time – I illustrated this problem with two figures. The first showed the expected distribution of floor space in a theoretical city and the second the expected distribution of traffic. The two requirements are in mutual conflict: central blocks rise skywards as central roads become congested (Figure 9).
The next exercise looks at this problem. Many downtown areas make use of small blocks established at a much earlier period. The city of Singapore is a good example in which the image of Manhattan has been adopted on sites so small that the footprints of the tower blocks are often less than 25% the area of a typical New York high-rise. This has led to an exciting skyline of elegant, but essentially grossly inefficient, buildings. Take again a theoretical model of a downtown: first with every one of 5 x 5 = 25 blocks filled by built forms eight floors high (25 x 8 = 200 units). Now suppose that this floor space is reconfigured to look more like the arrangement with which we are familiar: each annulus from the center has the same amount of floor space so that the 16 outer blocks have built forms four floors high (4 x 16 = 64 units); the next annulus of eight blocks retains the eight story high built forms (8 x 8 = 64 units), while the center 64 units of floor space (at 32 : 1 ratio) occupy just one block to make 192 units in all annuli. There is a discrepancy of 8 units between the two arrays, but this of no consequence to the argument. The “exponential” growth in building height mirrors, but exaggerates, reality. The same roads serve the buildings, but it is clear that the central area will suffer acute congestion.

The whole floor space may be reconfigured around a single courtyard. The surrounding road consumes the same amount of land as before, but now there are just four roads of much higher capacity than the twelve roads of the original scheme. There is a nine-fold reduction in intersections improving pedestrian flows, reducing the risk of accident and the pollution of waiting vehicles. The most surprising result is that much of the land of the narrow and traffic-vulnerable sidewalks is collected together in a protected, traffic-free, courtyard the size of nine former blocks. The building is seven floors high containing 196 units of floor space (Figure 10). Efficient tall buildings tend to have large footprints because of the service core. It is possible to retain the court form while halving the building depth for natural ventilation and lighting. This is achieved in a four court array.
Even so, for every two lanes of traffic per road in the original layout, the single court provides five lanes per road and the four courts provide three lanes per road: always holding the area of land devoted to transportation constant. The lane capacity of roads increases with the number of lanes. Keeping the road surface constant, suppose the twenty-five blocks are regrouped into four blocks, two-lane roads are replaced by three-lane roads with a lane capacity one and a quarter times higher, or into a single block surrounded by four-lane roads with a lane capacity just over double the two-lane scheme. Cerdá’s problem finds a solution in such a courtyard arrangement, and this is precisely the form he chose as a type for Barcelona.
These studies of built forms and the distinct ways they use land are based on the simple criterion of a sky ratio, or the angle at which the base of one built form makes with the parapet of a form immediately opposite. It will be observed that this angle is steep in the case of the pavilion forms, but much shallower in the comparable court forms.

The implication is that the ground floor occupant in the court forms are likely to see more sky than those in the tower configurations. A criticism that I make myself is that the sky view from a ground floor window is not as simple as the model assumes. In each individual case, a computer simulation will be needed to compute a more realistic assessment. The tower forms suffer from not taking into account the views around the sides of the built forms opposite, whereas the court only has internal views of the buildings surrounding the court (Figure 13).

A practical example of the effect of sky angles on architectural design is shown in the considerations that led to Leslie Martin’s project for new office buildings around courts in
Whitehall, London. Historically, the early Tudor streets in the area, with their overhangs, may be seen as an attempt to have light penetrate to the center line of each floor. The later post-Fire (1666) terraces solved this problem by varying the story heights. The Whitehall project proposed a stepped section on either side of an enclosed galleria for public use. The room depths now vary, but this variation accommodated the variety of room sizes in the program. Research on atrium offices since has confirmed that such a strategy has potential for energy saving in the British climate. The stepped-back elevations to the court provide a greater sense of openness in the courts themselves as well as in the offices on opposite sides.

Figure 14. Left, the sky angles for a typical Tudor street. Center, the sky angles for a typical post-Fire street. Right, the sky angles for the cross-section of the proposed Whitehall court form.

Mathematical processes of thought have been used here to provide counter-examples to the conventional wisdom that society and professions have a habit of promoting. Alfred North Whitehead has described this mode of thought as “speculative”; it is profoundly radical in that it attempts to go back to basics. The alternative to centralized development gained the name “perimeter development”. Some London boroughs made use of this alternate concept in their public housing, especially following a devastating gas explosion in one high-rise residential tower. The award-winning work of Richard McCormac is notable in this regard, as is a scheme in Battersea in which a proposal for twenty-two-story towers was replaced by four-story town houses and walk-up apartments. Later New Towns in the UK such as Livingstone and Milton Keynes adopted perimeter development. Richard McCormac realized that it was possible to increase the perimeter length around a site by something akin to crenelations, or more generally by foldings. This corresponds to increasing the available frontage for the homes.

Figure 15. The effective length of a perimeter may be increased by folds and crenelations. The numbers compare the increase to the straight line as 100 units.
Phoenix, Arizona, plans to expand urban development north of the city, into the Sonoran Desert for some 300,000 persons. The desert is of great ecological significance. The Nature Conservancy and partner organizations have identified some 100 landscape-scale conservation sites and some 30 smaller areas. Urban developments may flood around these conserved “islands”. The lines of desert cities that fan out from Palm Springs, California, for some 50 kms north and south of Joshua Tree National Monument provide an example of this open-space centered development. In opening a workshop on the Phoenix-Sonoran development, I questioned the drawing of closed boundaries around future urban uses. All parties agreed that people were migrating to the desert because they wanted to live the desert life, yet the planners and architects were visualizing gated and walled communities little different from other exurban developments elsewhere. It seemed to me that a desirable goal would be to maximize the boundary between the desert and the homes. I presented a simple fractal demonstration of how the urban area allocated might remain constant, but the perimeter might be increased three-fold over the closed boundary favored by surveyors, or as much as required depending on scale. This is not dissimilar to the convolutions of inlet and isthmus that developers create in new oceanside communities to maximize access from the homes to water. Curiously, and sympathetic to my prior arguments, the original square tends towards a cruciform after a few iterations of the crenelating rule.

![Fractal development](image)

**Figure 16. Fractal development in which the perimeter between one use and another is increased while holding area constant (fractal dimension 1.36)**

**Forms of beauty: qualitative studies**

For Froebel, forms of beauty were abstract designs that exhibited strong symmetries. Interestingly, he would break the symmetry of one design and, through a series of moves, transform the design into a novel one. Symmetry is unavoidable, in modern terms, since asymmetry, called the identity, acts like the “one” of multiplication in common arithmetic, and thus it counts as a unique form of symmetry. An object has symmetry if there are spatial transformations that allow the object to move, and yet end up occupying the initial space. For example, if I take a paper square and place a pin in the center, I can rotate the square through any multiple of the right angle and each time its position will coincide with the original. I could also turn the square over along any one of four axes through the center – the horizontal, vertical, and the two diagonal – to achieve the same
result. In the modern group theory of symmetry, it is possible to precisely say whether an object has more or less symmetry than another, and to know just how many subsymmetries of an object there may be to exploit in design. Louis I. Kahn’s Assembly building in Dacca, Bangladesh, provides an excellent lesson. The ceiling of the main chamber is based on a regular 16-gon (the number 16 being a symbol of wholeness in Indo-Arabic cultures). The order of symmetry of this shape is 32. According to Lagrange,\textsuperscript{14} the order of subsymmetries must be factors of this number: 16, 8, 4, 2 and 1, the identity. The facets of the ceiling mark out the symmetry of the octagon (order 16), the mosque is based on the symmetry of the square (order 8), the double squares of the offices follow the symmetry of the rectangle (order 4), while their internal planning is bilateral (order 2). Finally, the whole design has no overall symmetry (order 1): a globally asymmetrical composition that is replete with local symmetries. Quite deliberately, the powerful axis of the entrance lobbies, crossing that of the assembly chamber itself, is symbolically broken by the mosque, which adjusts its orientation to Mecca.

While symmetry may be described mathematically, it is not the conventional mathematics of solving numerical equations – typical of quantitative studies. The geometry of Euclid, for example, is about determining the measurements of lines, areas and volumes and comparing these properties for different figures such as the five Platonic solids. There is absolutely no explicit discussion in \textit{The Elements} of the symmetry of these forms, yet it is symmetry that provides their compelling, foundational importance today.\textsuperscript{15}

A simple example of qualitative study of form is given by examining the set of twelve pentominoes — two-dimensional figures in which five identical squares are joined in all possible ways edge to edge. Early configurational studies in architecture made use of such objects as synthetic, impersonal populations of designs. The twelve configurations may be grouped according to their symmetries: five are asymmetrical (identity, of order 1); one exhibits half-turn symmetry (the cyclic group, $C_2$, of order 2), two are bilaterally symmetrical in the orthogonal direction, two along the diagonal (all four are examples of the dihedral group, $D_1$, of order 2), one shows bilateral symmetry in two directions (the dihedral group, $D_2$, of order 4), and the regular cross has the full symmetry of the square (the dihedral group, $D_4$, of order 8). Preserving the orthogonal orientation, the identity forms may assume eight distinct positions; the $C_2$ form, four positions including left- and right-handed versions; the $D_1$ forms have four positions; the $D_2$ form two distinct positions, and the regular cross (say, the exterior of Palladio’s \textit{La Rotonda})\textsuperscript{16} has only one position.

The population of pentominoes may also be classified topologically in terms of connectivities between adjacent squares. A linear, planar graph represents this where a vertex of the graph stands for a square and the line joining two points marks a shared edge. The twelve designs fall under just four equivalence classes: seven are topologically equivalent to a simple “path”, three to a “tree” with two branches; one to a “tree” with three branches; and one with a “cycle” where four squares make mutual contact around a common point. Another representation is the planar map, which may be thought of a
rubber-sheet transformation of a pentomino itself. I shall return to the importance of this later.

Polyominoes may be used to define the footprint of a built form. Suppose the task is to accommodate eighteen cubic units. One built form, eighteen cubic units high, would achieve this. There is also just one way with a footprint of two units – a nine cubic unit high configuration. With a footprint of three units there are two distinct plan possibilities at 6 cubic units high: a simple slab, or an L-shape. With three floors, the footprint is six units. There are thirty-five hexominoes. Thirteen have the topology of a “path” – a simple string of six units on each floor, no matter how this is folded. Every cubic unit has at least two exposed sides. When the configurations have the topology of a “tree” with branches then some units will have only one exposed face, or possibly none. Twenty-two of the configurations are like this. A vertex with three branches means that that unit has $4 - 3 = 1$ exposed face. A vertex of degree 4 represents a unit with $4 - 4 = 0$ exposed faces. That is to say, the unit is completely surrounded on the interior. Eight of the scheme include a “cycle” in their topology. These are the most compact schemes. The mean distance measure within each floor of the configuration ranges from 1 for the nine-story slab; 1.33 for both six-story designs, and from 2.33 to 1.67 for the thirty-five three-story schemes.
Figure 18. Left, eighteen spatial units assembled in one nine-story tower based on the unique domino plan. Right, the two six-story towers generated from the two triomino plans. The connectivity graphs and mean graph distance are given for the configurations.

Figure 19. The population of three-story built forms based on the prismatic development of the 35 hexomino plans. The lightly tinted areas show spaces with only one external face, and the dark tint indicates a space that is entirely interior.
An indication of the combinatorial explosion with this approach is demonstrated by the 1285 distinct two-story 9-omino designs, and the 192,622,052 one-story 18-omino configurations. Almost all of these schemes are asymmetrical.

If the quantitative studies have the musty scent of Renaissance isoperimetric and proportional problems, then these particular qualitative studies exude the antiseptic smell of Durand’s polytechnic rationalism. Discrete elements are combined in tinker-toy fashion. Even if three-dimensional polycubes were used instead of the prismatic extension of two-dimensional polyominoes into three dimensions, such as in the above example, nothing will have changed – we are still in tinker-toy land. But until recently, this has been the cost of the abstraction necessary for the mathematization of synthetic design in architecture (Figure 20).

To generate more realistic schemes, several authors, including myself, toyed with the idea of the “dimensionless grating.”¹⁸ This is best illustrated with one of the hexominoes (Figure 21). It will be seen that it sits in 3 x 4 grid, or grating. This configuration may be described using a Boolean – 0, 1 – code for computational purposes:

1110 | 0011 | 0001

reading from left-top to bottom-right of the grating.¹⁹ Dimensions may then be applied to the grating to transform the configuration, and the grating may be “torn” to produce a yet more general transformation, while making the straight lines curvilinear would produce “morphs” similar to those found in D’Arcy Thompson’s *On Growth and Form*. Thompson’s book has been an influential source in the development of contemporary morphological studies in architecture.²⁰
Let me turn now to an example from my stay in Singapore. I had used the earlier arguments to suggest that Singapore’s largely high-rise, UN Habitat award-winning, New Towns had never been necessary. Those arguments had been introduced to at least one Singaporean architect, Tay Kheng Soon, who for his criticism found himself exiled in Malaysia planning and designing high-density, low-rise housing. My own contribution was deliberately contentious. I showed that Singaporeans could have been housed in one-story dwellings on no more urban land than currently used. The form of housing that achieved this result was the courtyard, which has a long tradition in the Chinese, Indian and Islamic cultures that meld together in Singaporean society, in contradistinction to the high-rise slab and tower of twentieth-century internationalism.

Here, I use the example to illustrate an application of the modern mathematics of symmetry. One type of site that was considered was the familiar nine-square pattern. The site was walled and the dwelling could occupy at most 5/9 of the land within. For the sake of argument, I envisaged five units somewhat after the open manner of Kahn’s Trenton bath-houses, leaving the enclosures to individual owners. The question was to maximize variety within these parameters. This reduces to the mathematical question: how many ways are there of “coloring” five of the squares in a 3 x 3 square. The colored squares would then represent a possible house plan and the uncolored squares would represent the open spaces, or courtyards. A simple answer would be the mathematical expression “9 choose 5” which is the factorial expression 9!/5!4! equal to 126. Many of these plans are the same under symmetry: right- and left-handed, or rotated. How many distinct plans is given by one of the most elegant pieces of modern combinatorial theory – George Pólya’s Enumeration Theorem. I can only hint at its power here. First, I give the answer diagrammatically (Figure 22).

Figure 22. The catalogue of the 126 courtyard houses in a 3x3 square in which four of the nine squares are open spaces. The twenty-three black figures are the configurations enumerated by Pólya’s theorem, the gray figures show symmetrically-equivalent configurations in the vertical columns. The indents show configurations in which all five units are fully connected, then those in which some contact is corner to corner, and finally configurations in which the house is divided in two or more parts across a courtyard.
The result that there are just 23 distinct configurations under symmetry is derived from the group of Pólya figures which indicate the symmetry operations on the nine squares. The central square always remains the central square. Corner squares may exchange only with corner squares, and the four side squares with side squares (Figure 23).

![Figure 23. The group of eight Pólya figures for the nine square problem. From left to right, if the nine-square is not moved, then all nine squares also remain stationary. "Nine 1-cycles" is symbolically represented by the expression below. Next, the figures show rotations through 90°, both clockwise and anticlockwise, to give "two figures with one 1-cycle and two 4-cycles". Center, is the exchange resulting from a half-turn, 180° rotation. This gives one figure with "one 1-cycle and four 2-cycles". Right, four figures in which reflection is indicated in the vertical axis, the horizontal, the leading and trailing diagonals. The symbolic notation reads "4 figures with three 1-cycles and three 2-cycles", the 1-cycles occurring along the axis of symmetry in each case.](image)

The information contained in the eight figures is captured in the cycle index:

\[
\frac{1}{8}(f_1^9 + 2f_4^2 + f_1f_2^4 + 4f_1^3f_2^3),
\]

which is short-hand for "in eight figures we may choose a figure with nine 1-cycles or 2 figures with a 1-cycle and two 4-cycles or a figure with a 1-cycle and four 2-cycles or 4 figures with three 1-cycles and three 2-cycles". The next step is to make the following substitutions:

\[
f_1 \rightarrow 1 + x, \quad f_2 \rightarrow f + x^2, \quad f_4 \rightarrow 1 + x^4,
\]

which are examples of what Pólya calls the “figure inventory”. The first basically reads “in a 1-cycle we may choose to either not to color an element, 1, or to color it, x”. The second reads “in a 2-cycle, if we color one element, then the other element of the pair must carry the same color – hence the term x^2”. The third reads “in a 4-cycle, if one element is colored then so must the other three be colored likewise under symmetry — hence x^4”. After the substitution, the cycle index looks like
which, when expanded, simplifies to the “counting polynomial”

\[ 1 + 3x + 8x^2 + 16x^3 + 23x^4 + 23x^5 + 16x^6 + 8x^7 + 3x^8 + x^9. \]

The powers of \( x \) represent those designs with that number of colored elements. There is one configuration with no colored elements \( (1 = x^0) \), as there is just one in which all nine elements are colored \( (x^9) \). The number we are looking for is the coefficient of \( x^5 \) which tells us there are 23 distinct designs of the nine square homes under symmetry. These are the ones illustrated in black above. The coefficient of \( x^4 \) is the same, since this could be read as coloring the four open spaces instead of the five enclosures.

An elaboration of this example includes a second color that stands for the service unit to the served spaces. The inventory then looks like

\[ f_1 \rightarrow 1 + x + y, \]

and so on for the 2-cycles and 4-cycles. The coefficient of the term \( x^4y \) counts the number of configurations with four served spaces, \( x^4 \), and one service space, \( y \). That number is 89 (Figure 24).

![Fig 24](image)

Figure 24. Left, a group of nine-square homes arranged around a common open space. The darker tint indicates a service area. Center, the *poché* of the columns creates its own aleatoric pattern, although strictly based on symmetry considerations. Right, the *poché* of columns in Kahn’s project for the Adler House, 1954-5

Finally, in this section on qualitative studies, I return to the representation of architectural plans by planar maps. It turns out that the number of potential architectural plans can be enumerated and classified. There are only a certain number of planar maps with \( n \)-faces, or the partition of the plane into \( n \)-rooms. These in turn may be derived
from trivalent planar maps through “ornamentation”, and every trivalent map in the plane is a stereographic projection of a three-dimensional trivalent polytopes. In other words, Plato had the right idea.

Christopher Earl and I wrote,

_Essentially, the catalogue of trivalent 3-polytopes constitutes the “periodic table” for the “chemistry” of room formations. The catalogue is exhaustive. Such forms pre-exist in the recursive sense. They are produced by a simple rule system. At root we are saying that room formation is not itself a design problem, whereas ornamentation is. Immanent structures for each and every room formation are finite in number and are known aprioristically: architectural design is pre-eminently a matter of selection and the appropriate physical and material transformation of one of these fundamental plans.”_

Figure 25. Top left, the three necessary and sufficient rules to enumerate all trivalent 3-polytopes starting with the tetrahedron. The first rule shaves a 3-vertex to create a new triangular face, three new lines, and two new points; the second cuts a line to produce a rectangular face, three new lines and two new points; and the third takes three contiguous vertices and makes a cut which produces a pentagonal face. Starting with a four-faced tetrahedron, the next trivalent polytope, like a slice of cheese, has five faces; then there are two distinct six-faced polytopes including the cube; and five trivalent polytopes with seven faces. Right, stereographic projections through faces of the polytopes produce a countable number of trivalent planar maps, in this case with seven faces including the exterior.

It is also a mathematical fact that every such trivalent map can be represented by an orthogonal arrangement of lines – that “free-forms” are no freer than the right-angle. It may be of interest to note that there are just three “perfect” plans corresponding to the tetrahedron, the cube and the dodecahedron. In the first, all four spaces connect to three others; in the second, all six spaces connect to four others; and in the third, all twelve space connect to five others (including the exterior in all cases).
Figure 26. Perfect plans. Left, the planar maps of the tetrahedron, the cube and the dodecahedron. Center, orthogonal presentation of these maps as rectangular dissections. Right, free-forms satisfying the same topologies

**Forms of life – relations**

We leave behind classical and neoclassical approaches such as those I have described in the previous two sections with the statement by George Stiny that “a design is an element in a n-ary relation among drawings, other kinds of descriptions, and correlative devices”, and that “a relation containing designs is defined recursively in an algebra that is the Cartesian product of other algebras”. This is the language of modern mathematics and computation, not – unfortunately – of contemporary computing such as to be found in commercially available computer-aided design software. I draw a strong distinction between “computation” and “computing”. Designers who compute using certain programs unwittingly become prisoners of encoded modes of thought, which, despite the flashiness of three-dimensional display, remain one-dimensional in origin. Almost all current systems rely on representations of lines, planes and solids as lattice point sets. Except for the square, it is rarely appreciated that a computer cannot draw an equilateral triangle, or a regular pentagon, or any other regular polygon. It has to fudge and fool the user into suggesting it can. Nor can a computer scale up a square to double, or triple its area, and draw it. It looks as if it is capable, but the task is actually beyond its digital powers. Digitization has its terrible limits, and it exacts a frightening intellectual price.

The shape grammar formalism provides an escape in its approach to shape computation. I have no space here to go into details. It is perhaps worth remarking that the Turing Machine is the standard theoretical construct for modern computation, and that a shape computation satisfies the criteria for a Turing Machine using shapes in place of symbol strings, or other tinker-toy sets in which there are primitives, or fundamental units. I will say that the repercussions of “seeing” shapes in a calculation liberates us from the norms of thinking with symbols, words and numbers.

Briefly, I will mention three examples of shape computation that the reader may care to follow up at more leisure. First, the shape grammar formalism has been successfully applied in examining questions of style in architecture and the visual arts. Some of the best work in this area is by Terry Knight, whose examination of the transformation in style for Frank Lloyd Wright’s Prairie houses to the Usonians is particularly revealing. My second case is recently completed work by Dr. José Pinto Duarte at MIT, who wrote a shape grammar interpreter to emulate Álvaro Siza’s housing at Malagueira, Portugal. The 1,200 home project started in 1977 and continues today. Duarte has enjoyed the support of Siza in this exercise. Finally, anyone who has had the misfortune to see a
Frank Gehry building under construction, before it is dressed in its sexy metallic integument, will have been gravely disappointed by the massive clumsiness of the angled standard framing. Skin and skeleton are no more integrated than in Eiffel’s Statue of Liberty. At the Engineering Design Centre, University of Cambridge, Dr. Kristina Shea is using the shape grammar formalism to take up the “next challenge in free-form structural design”, which she sees as the “simultaneous design of intriguing surfaces and their corresponding structure”.

Concluding remarks

In the late 1960s, I was told that the architects Alison and Peter Smithson found themselves at a loss with their son’s schoolwork involving the “new maths”. Apparently, they brought this generational gap problem to the attention of the Royal Institute of British Architects (RIBA) Library Committee, which in turn invited me to write a book for architects that would illustrate the potential of the “new maths” in their field. I invited Philip Steadman to join me in this task. The Geometry of Environment was published by the RIBA in 1971, and subsequently by Methuen and the MIT Press. Hungarian and Italian editions followed. The Smithsons had been impressed earlier by Rudolf Wittkower’s Architectural Principles in the Age of Humanism (1949) and, in the debate on that subject at the RIBA, Peter Smithson had expressed an interest in the relationship of mathematics to architecture at mid-twentieth century as a parallel to the Renaissance. I have not mentioned my own Architectonics of Humanism, published in 1999 as a fiftieth anniversary companion to Wittkower, although its subject is probably closer to current interests among Nexus readers than the matters I have described. I must confess that my main motivation in writing Architectonics was not the proportional analysis of architectural works, but the origins of the modern arithmetization of geometry, essentially the digitization of shape. There is no map, there are no shapes, in Alberti’s Discriptio Urbis Romae, only lists of coordinates – it is pure digitization. The primacy of number has dogged us architects ever since, and shape has been put on the back-burner. Yet, surely, giving shape to the human environment is the architect’s primary task. The craze for digital imaging only serves to obscure centuries of neglect in the study of shape itself.

In The Geometry of Environment, Philip Steadman and I introduced to the architectural profession and schools the mathematical concepts of relations and mappings; set theory and Boolean algebra; group theory and symmetry; spatial transformations and matrix representation; and graph theory. We also updated proportional theory to cover modular co-ordination. Today, shape grammars use an extension of Boolean algebra called a Stone algebra that allows for a null shape, but not for an infinite shape; they also employ Euclidean spatial transformations that include scaling, translations, rotations and reflections. Symmetry plays a key role in local shape rule applications. Graph theory provides another description. Lattice theory provides the means to partially order decompositions of shape, which with each shape rule application may change – there are no fixed parts, only creative ambiguities. We can know what constitutes a shape after the
final computation in its generation, and even then it depends on what rules we adopt to view it!

I have not discussed space syntax, which has proved to be popular in South American countries and especially in Brazil. Its recent architectural applications include predicting pedestrian movements in buildings and the visual comprehension of interior spaces (isovist studies). The architect Norman Foster has employed these techniques in some projects.

It would be amiss not to mention my old Cambridge student companion, Christopher Alexander, and the stimulus that his 1964 Notes Towards a Synthesis of Form gave to the mathematical treatment of architectural topics. Even though he came to reject his original thesis, the subject of decomposing complex programmes into manageable sub-problems persists. My problem in writing on mathematics and architecture since 1960 remains the utter unmanageability of the topic. In this, the Queen of the Sciences has forsaken me.

Notes

1. First in architecture at Cambridge University, for two years in urban studies at Harvard and MIT, back at Cambridge as Director of the Centre for Land Use and Built Form Studies, then in systems engineering at the University of Waterloo, in design technology at The Open University, later at the Royal College of Art, London, and – for the last two decades – in architecture and urban design at the University of California, Los Angeles, with a six-month stop-over as a consultant at The National University of Singapore.

2. I dedicate this paper to the memory of Dr John Ashby (1928-1999), a biochemist, and who, as a publisher with Pion Limited, London, had the foresight and courage to nurture and promote such a journal.

3. For example, I co-authored a paper with the graph theorists Frank Harary in the USA and R W Robinson in Australia in 1978 entirely by correspondence. I met Harary some years later at a cocktail party in Cambridge, but Robinson, never.

4. Professor of Architecture, University of Cambridge. Best known in Portugal for his work at the Gulbenkian Foundation in Lisbon, but more widely for the Royal Festival Hall, London.

5. Friedrich Froebel (1782-1852) pioneered the modern study of form. He employed the three Aristotelian categories of quantity, quality and relation in structuring the educational content of his “gifts” which ran the gamut from solid, plane, line to point in descending order of concreteness, and ascending order of abstraction. His categories were “forms of knowledge” in which quantitative aspects are studied; “forms of beauty” in which the qualities of spatial transformations and symmetries come into play; and “forms of life” in which the forms are related to actual objects – a house, a bath, a chair, and so on. I use these categories to organize this paper.

6. Engineer Sr. Luis Lobato was an enthusiastic protagonist of this work.


9. The unit of distance taken in this example is half a block length. The mathematical question of mean distances is addressed in Jenny A Baglivo and Jack E Graver, Incidence and
An architectural investigation of built forms is found in Philip Tabor, "Traffic in Buildings," Ph.D. Diss., University of Cambridge, 1971. On the assumption that all trips are equally likely, it should be noted that the compact city-section favors short trips over the other two configurations, whereas the annular form has an even distribution. In practice, trips are not likely to be equiprobable, and this needs to be factored in according to expectations.

10. For a practical application of these ideas to the Central Region of Chile, see Marcial Echenique “'Let's build in lines' revisited,” Planning and Design 21, 1994: s95-s105.

11. "Ruralize the urban, urbanize the rural…fill the earth." I was first introduced to the works of Ildefons Cerdà by Dr. Marcial Echenique who had joined LUBFS from Barcelona to direct the urban systems study. A useful summary is Cerdà: The Five Bases of the General Theory of Urbanization, Arturo Soria y Puig, ed. (Madrid: Electa Espana, 1999).

12. The Sonoran Desert covers about 222,700 sq. km. in California and Arizona in the United States, and Baja California and Sonora in Mexico. It is the subject of a case study by the Nature Conservancy in the AAAS Atlas, pp 188-89. In 1995 the regional population was 5.5 million, growing at the rate of 3.0 per cent per year.

13. Froebel had trained as an architect in Frankfurt, but became an assistant to Christian Samuel Weiss, one of the founders of the modern science of crystallography.

14. Joseph-Louis Lagrange (1736-1813) was the founding Professor of Mathematics at L’Ecole Polytechnique, Paris, during the tenure of the architect J.N.L Durand in the stereotomy department under Abbé Haüy – Weiss’s French rival in the architecture of crystal forms.

15. There are just seven non-infinite spherical groups in space, two each associated with the Platonic duals, the cube/octahedron and the dodecahedron/icosahedron, and three related to the self-dual tetrahedron. See, for example, Jenny A Baglivo and Jack E Graver, Incidence and Symmetry in Design and Architecture, pp. 246-250.

16. The interior of La Rotonda has the subsymmetry $D_2$.


18. This concept arose initially from M.H.A Newman, Elements of the Topology of Plane Sets of Points (Cambridge: Cambridge University Press, 1964). Newman had directed the effort to build the world’s first programmable, electronic computer at the University of Manchester, 21 June 1948. He is especially regarded for having recruited Alan Turing – of Turing Machine fame – to his staff.


21. “Meng Ker introduced me to the book – Lionel March, Urban Space and Structure – and from there we did some morphological studies together and he did his thesis on Woodlands, one neighbourhood of Woodlands. And he redesigned a neighbourhood, the typical HDB [Housing Development Board] neighbourhood which was 10, 12, storeys at that time. He redesigned it to 5, 6 storeys, with one-third of the dwellings having gardens. So, to answer your question, are there alternatives to HDB, there are plenty of alternatives.” Interview by

23. George Pólya, Robert E. Tarjan, Donald R Woods, *Notes on Introductory Combinatorics* (Boston: Birkhäuser, 1983). In these notes, the lively "voice" of the nonagenarian Pólya can be heard still teaching at Stanford University, California.


29. This work is available on line at http://destec.mit.edu/malag/. Stiny and Knight were among Duarte’s Ph.D. advisors.


31. The work of Dr. Mário Júlio Teixeira Krüger, University of Coimbra, relates the space syntax tradition with the LUBFS approach. Dr. Nuno Portas was an important link with the Cambridge school following the 1974 Portuguese Revolution, as was Manuel Solá-Morales Rubó in Barcelona, who steered through the 1975 Spanish edition of *Urban Space and Structures*.
